Mirror symmetry between Landau-Ginzburg models

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mirror symmetry

Calabi-Yau moduli space

LG point

Landau-Ginzburg A-model (FJRW-theory)

mirror
In this talk we would like to discuss LG models (closed string), and mirror symmetry identifies two nontrivial theories of singularities. This is joint work with W. He, Y. Shen and R. Webb.
Our main result can be summarized as

**Theorem (He-L-Shen-Webb)**

For a general class of mirror singularities $W$ and $W^T$, the associated quantum intersection theories are equivalent under mirror transformation.

This is the parallel of the mirror theorem for CY manifolds (Givental and Lian-Liu-Yau) to the setting of LG model/singularity theory.
LG model is associated to a holomorphic function

\[ f : X \rightarrow \mathbb{C}. \]

We will focus on \( f : X = \mathbb{C}^n \rightarrow \mathbb{C} \), where \( f \) is a weighted-homogeneous polynomial of weights \( q_1, \ldots, q_n \):

\[ f(\lambda^q x_i) = \lambda f(x_i), \quad \forall \lambda \in \mathbb{C}^*. \]

**Definition**

The central charge is defined by

\[ \hat{c}_f = \sum_i (1 - 2q_i) \]

The central charge \( \hat{c}_f \) plays the role of dimension for Calabi-Yau models.
The miniversal deformation space $Def(f)$ of $f$ is locally parametrized by

$$\text{Jac}(f) := \mathbb{C}[x_i]/(\partial_i f).$$

If we choose a basis of $\text{Jac}(f)$ by weighted homogeneous polynomials $\phi_\alpha(x)$, then the deformation can be represented by

$$F(x, u) = f(x) + \sum_{\alpha=1}^{\mu} u_\alpha \phi_\alpha(x)$$

where $\mu$ is the Milnor number of $f$. 
Around 1979-1982, Kyoji Saito developed a period map theory for $F$ which induces a flat structure (Frobenius manifold structure after Dubrovin) on $\text{Def}(f)$.

Roughly, Saito constructed a special family of volume forms (which is called a primitive form)

$$\Omega_u = \frac{dx_1 \wedge \cdots \wedge dx_n}{P(u, x)}, \quad \text{where } P|_{u=0} = 1,$$

as a deformation of the CY volume form $\Omega_0 = dx_1 \wedge \cdots \wedge dx_n$, and look at the oscillatory integral

$$\int_{\Gamma} e^{F/z} \Omega_u.$$
Saito’s primitive form

The primitive form is chosen in a remarkable way that there exists a change of coordinates (flat coordinates) such that

$$\tau_i = \tau_i(u)$$

- the oscillatory integral satisfies the following equation

$$\left( \partial_{\tau_a} \partial_{\tau_b} - z^{-1} \sum_c A^c_{ab}(\tau) \partial_{\tau_c} \right) \int_\Gamma e^{F/z} \Omega_u = 0.$$ 

- there exists constant matrix $\eta_{ab}$ (given by residue pairing on $\text{Jac}(f)$) and an analytic function $F_0(\tau)$ such that

$$\partial_{\tau_a} \partial_{\tau_b} \partial_{\tau_c} F_0(\tau) = \sum_d A^d_{ab}(\tau) \eta_{cd}$$

- $\{A^c_{ab}(\tau)\}$ defines the structure of an associative algebra. (WDVV equation in modern language).
$\mathcal{F}_0(\tau)$ (potential function) is the mathematical counterpart of the genus 0 generating function of Landau-Ginzburg B-model. In the LG case, there is Givental’s higher genus formula available. The whole package is sometimes referred as Saito-Givental theory.

This construction is generalized to arbitrary isolated singularity (not necessarily weighted homogeneous) by Morihiko Saito and certain Laurent polynomials by Douai-Sabbah, etc.
Some examples

For weighted homogeneous singularities, only known expressions of primitive forms are (K. Saito 1982)

- ADE ($\hat{c}_f < 1$): $P \equiv 1$.
  \[ \Omega_u = d^n x \]

- Simple elliptic singularity ($\hat{c}_f = 1$): $P$ is a period of the elliptic curve defined by marginal deformation.
  \[ \Omega_u = \frac{d^n x}{\text{Period}}. \]

- $\hat{c}_f > 1$. Closed formula doesn’t exist in general.
The space of primitive forms can be identified with the choices of the so-called **good basis**. Let

$$\eta_f : \frac{\Omega^n}{df \wedge \Omega^{n-1}} \times \frac{\Omega^n}{df \wedge \Omega^{n-1}} \to \mathbb{C}$$

be the residue pairing. This is naturally lifted to a pairing

$$K_f : \frac{\Omega^n[[z]]}{(zd + df \wedge) \Omega^{n-1}[[z]]} \times \frac{\Omega^n[[-z]]}{(-zd + df \wedge) \Omega^{n-1}[[z]]} \to z^n \mathbb{C}[[z]]$$

such that

$$K_f = K^{(0)} z^n + K^{(1)} z^{n+1} + \cdots$$

then $K^{(0)} = \eta_f$.

This is Saito’s higher residue pairing.
Good basis

Definition

For weighted-homogenous polynomials, a good basis is a monomial representative \( \{ \phi^\alpha(x) \} \) of a basis of \( \text{Jac}(f) \) satisfying the higher residue vanishing condition

\[
K(\phi^\alpha d^n x, \phi^\beta d^n y) = z^n \eta_f(\phi^\alpha d^n x, \phi^\beta d^n x), \quad 1 \leq \alpha, \beta \leq \mu,
\]

i.e. only the leading term of ordinary residue is possibly nonzero.

Good basis is generally not unique. Different choices of good basis leads to different primitive forms, hence different generating function of LG B-model. Later we will see that mirror symmetry favors for a particular choice.
From good basis to primitive form

Let \( \{ \phi_1, \cdots, \phi_\mu \} \) be a chosen good basis. The primitive form \( \Omega_u \) is uniquely determined by the pole condition

\[
\int_\Gamma e^{F(x,u)/z} \Omega_u = \int_\Gamma e^{f/z} d^n x + \sum_{\alpha} B_\alpha(u,z) \int_\Gamma e^{f/z} \phi_\alpha d^n x
\]

where

\[
B_\alpha = O(z^{-1}).
\]

Then

- coefficient of \( z^{-1} \) \( \implies \) flat coordinates
- coefficient of \( z^{-2} \) \( \implies \) first derivative of \( \mathcal{F}_0 \).

This equation leads to an algebraic recursive formula for perturbative expansion of the primitive form (Li-L-Saito).
Relation with Kodaira-Spencer gauge theory

[Bershadsky-Cecotti-Ooguri-Vafa]: B-model on CY three-fold can be described by a gauge theory

→ Kodaira-Spencer gauge theory.

The classical physics of Kodaira-Spencer gauge theory describes the deformation of complex structures on Calabi-Yau manifold.

[Costello-L]: This gauge theory is extended to arbitrary dimensional CY with gravitational descendants turned on, which we will call BCOV theory. Quantization via quantum master equation gives the higher genus B-model.
Let $X$ be a compact CY. Consider the smooth polyvector fields

$$PV(X) := \bigoplus_{i,j} A^{0,j}(X, \wedge^i T_X).$$

We can identify it with differential forms with the help of holomorphic volume form $\Omega_X$:

$$PV(X) \leftrightarrow \mathcal{A}(X)$$

$$(\partial, \bar{\partial}) \leftrightarrow (\partial, \bar{\partial})$$

It also allows us to define the trace map

$$\text{Tr} : PV(X) \to \mathbb{C}, \quad \mu \mapsto \int_X (\mu \uparrow \Omega_X) \wedge \Omega_X$$
BCOV theory on CY

The field content of our BCOV theory concerns the following differential graded complex

\[(\text{PV}(X)[[z]], Q), \quad Q = \bar{\partial} + z\partial,\]

where \(z\) is a formal variable representing the gravitational descendants.

The interaction term is given by the following local functional

\[I^{BCOV}(\mu) := \text{Tr} \langle e^\mu \rangle_0\]

where

\[\left< z^{k_1} \mu_1, \ldots, z^{k_n} \mu_n \right>_0 := \binom{n-3}{k_1, \ldots, k_n} \mu_1 \wedge \cdots \wedge \mu_n,\]

where \(\mu_i \in \text{PV}(X)\).
$I^{BCOV}$ satisfies the classical master equation

$$QI^{BCOV} + \frac{1}{2} \left\{ I^{BCOV}, I^{BCOV} \right\} = 0.$$ 

The nilpotent operator $Q + \left\{ I^{BCOV}, - \right\}$ represents the gauge symmetry in the BV-formalism. Equivalently $Q + \left\{ I^{BCOV}, - \right\}$ defines a $L_\infty$-structure on $PV(X)[[z]]$.

[Costello-L] This $L_\infty$-structure is quasi-isomorphic to the differential graded Lie algebra structure $(Q, [\cdot, \cdot], s)$ on $PV(X)[[z]]$. 


The genus 0 generating function of BCOV theory is given by tree level Feynman integrals

\[ \mathcal{F}_0 = \sum_{\Gamma: \text{Tree}} W_{\Gamma}(G, I^{BCOV}) \]

where \( W_{\Gamma}(G, I^{BCOV}) \) is the Feynman graph integral with

- Propagator: \( G \) is the kernel of \( \frac{\bar{\partial}^* \partial}{\Delta} \), where \( \Delta = \bar{\partial} \partial^* + \bar{\partial}^* \partial \).
- Vertex: \( I^{BCOV} \).
- External input:

\[ H^*(PV(X)[[z]], Q) \sim H^*(X, \wedge^* T_X)[[z]] \]

with a choice of splitting Hodge filtration.
BCOV theory can be twisted to LG models:

\[ f : X = \mathbb{C}^n \to \mathbb{C} \]

as before. The field content is replaced by

\[ \text{PV}_c(X)[[z]], Q_f := \bar{\partial}_f + z\partial, \]

where \(\bar{\partial}_f := \bar{\partial} + df \perp\). The interaction \(I^{BCOV}\) is as before, satisfying the twisted CME

\[ Q_f I^{BCOV} + \frac{1}{2}\{I^{BCOV}, I^{BCOV}\} = 0. \]

I will call this gauge theory the **LG-twisted BCOV** theory.
The tree level gives a generating function on

\[ H^*(PV_c(X)[[z]], Q_f) \simeq \frac{\Omega^n[[z]]}{(zd + df \wedge) \Omega^{n-1}[[z]]}. \]

A choice of good basis gives an identification

\[ H^*(PV_c(X)[[z]], Q_f) \simeq \text{Jac}(f)[[z]]. \]

The generating function leads to the prepotential given by Saito’s primitive form (Li-L-Saito+ a version of homological perturbation).
Roughly speaking, the usual GW-theory counts the solution of

\[ \{ \bar{\partial} \phi = 0 \mid \phi : (\Sigma_g, p_1, \cdots, p_k) \to (X, \gamma_1, \cdots, \gamma_k) \} \]

with boundary conditions \( \phi(p_i) \in \gamma_i \in H_*(X) \). This defines the quantum intersection theory on \( H^*(X) \).

LG twisting: let \( W \) be a weighted homogenous polynomial, \( G \) a group of diagonal symmetries of \( W \) (containing the obvious element generating the weighted homogeneity).
Fan, Jarvis and Ruan have developed a quantum intersection theory on (orbifold) Milnor ring by counting the solutions of the Witten equation on Riemann surfaces

\[ \phi : \Sigma \to \mathbb{C}^n, \quad \bar{\partial} \phi_i + \frac{\partial W(\phi)}{\partial \phi_i} = 0 \]

with boundary conditions on marked point specified by Lefschetz thimbles. This is called FJRW-theory, the LG analogue of GW-theory.

An algebraic formulation of FJRW-theory (narrow sector) is developed by H-L. Chang, J. Li and W-P. Li. (For r-spin, also Polishchuk-Vaintrob, Chiodo)
Berglund-Hübsch and Krawitz propose the mirror transformation of (invertible) LG polynomials together with orbifold groups

\[(W, G) \leftrightarrow (W^T, G^T)\]

Here $W$ is of the form

\[
W = \sum_{i=1}^{N} \prod_{j=1}^{N} x_{ij}^{a_{ij}}.
\]

We denote its exponent matrix by $E_W = (a_{ij})$. The mirror polynomial of $W$ is

\[
W^T = \sum_{i=1}^{N} \prod_{j=1}^{N} x_{ij}^{a_{ji}},
\]

i.e., the exponent matrix $E_{W^T}$ of the mirror polynomial is the transpose matrix of $E_W$. 
Example

\[ W = x^3 + xy^5 \] with exponent \((3 \ 0\ 1 5)\). Then the mirror is given by

\[ W^T = x^3y + y^5 \]

whose exponent is \((3 \ 1 \ 0 5)\).

If \( G = G^{\text{max}} \) the maximal symmetry group, then \( G^T = \{1\} \).
Kreuzer and Skarke have classified such singularities: it is a direct sum of the three following atomic types \((a \geq 2, a_i \geq 2)\):

- **Fermat**: \(x^a\).
- **Chain**: \(x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}\).
- **Loop**: \(x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_N^{a_N}x_1\).
Conjecture (LG-LG mirror symmetry)

FJRW theory of $(W, G^{\text{max}})$ (LG A-model) is equivalent to Saito-Givental theory of $W^T$ (LG B-model) with an appropriate choice of primitive form.

Some known results before:

- ADE ($\hat{c}_f < 1$): Fan-Jarvis-Ruan. (also Faber-Shadrin-Zvonkine, Jarvis-Kimura-Vaintrob for A-types)
- Simple elliptic ($\hat{c}_f = 1$): Krawitz-Milanov-Shen.

A basic obstacle for $\hat{c}_W > 1$ is the lack of knowledge on primitive forms. There is

- Exceptional unimodular singularities (which have $1 < \hat{c}_f < 2$): Li-L-Saito-Shen.
The favored good basis

Theorem (He-L-Shen-Webb)

Let $f$ be an invertible polynomial of atomic types. Then the following choice $\{\phi_\alpha\}$ is a good basis of $f$.

- Let $f = x^a$ be a Fermat, then $\{\phi_\alpha\} = \{x^r \mid 0 \leq r \leq a - 2\}$.
- Let $f = x_1^{a_1} + x_1x_2^{a_2} + \cdots + x_{N-1}x_N^{a_N}$ be a chain, then

$$\{\phi_\alpha\} = \left\{ \prod_{i=1}^{N} x_i^{r_i} \right\}_{r}$$

where $r = (r_1, \cdots, r_N)$ with $r_i \leq a_i - 1$ for all $i$ and $r$ is not of the form $(\ast, \cdots, \ast, k, a_{N-2l} - 1, \cdots, 0, a_{N-2} - 1, 0, a_N - 1)$ with $k \geq 1$.

- Let $f = x_1^{a_1}x_N + x_1x_2^{a_2} + \cdots + x_{N-1}x_N^{a_N}$ be a loop, then

$$\{\phi_\alpha\} = \left\{ \prod_{i=1}^{N} x_i^{r_i} \left| 0 \leq r_i < a_i \right. \right\}.$$
A Mirror Theorem between LG models

Theorem (He-L-Shen-Webb)

Assume $W$ contains no chain variables of weight $1/2$. Then LG mirror conjecture is true: the FJRW theory of $(W, G_W)$ is equivalent to Saito-Givental theory of $W^T$ at all genera with the choice of the favored good basis.

A minor situation for chain types with weight $1/2$ not covered, i.e., $W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}$ with $a_N = 2$. This is a technical difficulty of missing information about certain FJRW 3-point functions due to the non-algebraic nature of FJRW theory.
Sketch of proof:

1. Using reconstruction techniques from $g = 0$ to higher genus [Teleman, Milanov], we reduce to proving $g = 0$ only.

2. [A Strong Reconstruction Theorem]: the genus zero invariants are completely determined by 2-point, 3-point and 4-point functions accompanied with WDVV equation, String equation, Dimension Axiom and Integer Degree Axiom.

3. Compute explicitly 3-point and 4-point functions both on FJRW-side (via orbifold-Grothendieck-Riemann-Roch calculations) and primitive-forms (via Li-L-Saito recursive formula). We show that they match.
Thank You!