Superstring Compactifications To All Orders In $\alpha'$

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General Remarks

A vacuum of type II supergravity is

\[ \mathbb{R}^{1,2} \times M \]

- Minkowski space
- Compact 7d manifold \( R_{ab} = 0 \)

Supersymmetry in space-time \( \nabla_a \eta = 0 \)

The metric on \( M \) has \( G_2 \) holonomy...

Calabi-Yau, \( Spin(7) \) manifold
Can the classical metric of $G_2$ holonomy be modified to compensate the $\alpha'$ corrections to the equations of motion and supersymmetry transformations?

Supergravity is only the low-energy limit of string theory (or M-theory). We expect an infinite series of corrections to the two-derivative action which are suppressed at large volume...
I. Type II string theory on $G_2$ manifolds
   a) General remarks.
   b) Leading order correction.
   c) All orders in $\alpha'$. 

II. Interpretation in 3d and 4d field theory.

III. Massive Kaluza-Klein modes in superspace.
I. Type II String Theory On $G_2$ Manifolds
Tools

Given a 7d spin manifold $M$ there is a unit spinor $\eta$

$$\varphi_{abc} = \eta^T \Gamma_{abc} \eta$$

and a 4-form...

$$\psi_{abcd} = \eta^T \Gamma_{abcd} \eta$$

...related by

$$\psi = \ast \varphi$$

$$g_{ab} = g_{ab} [\varphi] = (\det s)^{-1/9} s_{ab}$$

$$s_{ab} = -\frac{1}{144} \varphi_{mn} \varphi_{bpq} \varphi_{rst} \varepsilon^{mnpqrs}$$
Leading Order Correction
**Gravitino supersymmetry transformation**

\[
\delta \psi_a = \nabla_a \eta + A_a \eta + i B_a^b \Gamma_b \eta
\]

- In 7d dimensions spinors have 8 real components.
  A basis is \( \{ \eta, \Gamma_a \eta \} \), \( a=1,\ldots,7 \).

- The coefficients are tensors
  \( A_a = A_a[\varphi] \quad B_a^b = B_a^b[\varphi] \)

At \( O(\alpha'^3) \)
\[
A_a = 0
\]

\[
B_a^b = \alpha'^3 \varphi_{acd} \nabla^c \left( \frac{1}{32g} \varepsilon^{dc_1\ldots c_6} \varepsilon^{bd_1\ldots d_6} R_{c_1c_2d_1d_2} R_{c_3c_4d_3d_4} R_{c_5c_6d_5d_6} \right)
\]
Supersymmetric Vacuum

To order $\alpha'^3$

$$\delta\psi_a = \nabla'_a \eta' + A_a [\phi] \eta + i B^b_a [\phi] \Gamma_b \eta = 0$$

- Primed quantities include corrections: $\eta' = \eta + O(\alpha'^3)$
- $\phi, \eta$ of $G_2$ holonomy manifold

Set up PDE

$$d\phi' = \alpha [\phi]$$
$$d\psi' = \beta [\phi]$$
$$\psi' = \ast \phi'$$

$$\alpha_{abcd} = 8 A_{[a \phi_{bcd}]} - 8 B_{[a {^e}\psi_{bcd}]^e}$$
$$\beta_{abcde} = 10 A_{[a \psi_{bcde}]} - 40 B_{[ab \phi_{cde}]}$$
Solving the first equation is easy since $d\varphi' = \alpha = d\chi$

$$\varphi' = \varphi + \chi + db$$

...while solving the second is more complicated

$$d\psi' = d\ast'\varphi' = \beta = d\xi$$

linearizing

$$\ast'\varphi' = \ast\varphi + \ast\left(\frac{4}{3} \pi_1 + \pi_7 - \pi_{27}\right)(\chi + db)$$

$$\Lambda^3 \cong \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3$$
PDE

\[ d \times \left( \frac{4}{3} \pi_1 + \pi_7 - \pi_2 \right) (\xi + db) = d \xi \]

\[ \chi_{abc} = -c \varphi_{abc} Z + 3c \varphi_{[ab} d Z_{c]} d \]

\[ \xi_{abcd} = -4c \psi_{[abc} e Z_{d]} e \]

\[ Z^{ab} = \frac{1}{32 g} \varepsilon^{ac_1 \ldots c_6} \varepsilon^{bd_1 \ldots d_6} R_{c_1 c_2 d_1 d_2} R_{c_3 c_4 d_3 d_4} R_{c_5 c_6 d_5 d_6} \]

known
PDE

\[
d \ast \left( \frac{4}{3} \pi_1 + \pi_7 - \pi_{27} \right) (\chi + db) = d \xi
\]

Not very illuminating! To make progress decompose

\[
b \in \Lambda^2 \cong \Lambda^2_7 \oplus \Lambda^2_{14}
\]

Any \( b \in \Lambda^2_7 \) is a diffeomorphism

Take \( b \in \Lambda^2_{14} \) and \( d^+ b = 0 \)
Laplace Equation

\[ \Delta b = J \]

\( b \in \Lambda^2_{14} \)

\( d^\dagger b = 0 \)

\[ J = d^\dagger \rho \]

\[ \rho = -\star \xi - \left( \pi_{27} - \pi_7 - \frac{4}{3} \pi_1 \right) \chi \]

Facts: \( J \) is orthogonal to harmonic forms and \( J \in \Lambda^2_{14} \)

The Laplace equation is solvable!
All Orders In $\alpha'$
Gravitino supersymmetry transformation

\[ \delta \psi_a = \nabla'_a \eta' + A_a \left[ \varphi' \right] \eta' + iB_a^b \left[ \varphi' \right] \Gamma_b \eta' = 0 \]

Tensors

\[ \varphi'_{abc} = \eta'^T \Gamma'_{abc} \eta' \]
\[ \psi'_{abcd} = \eta'^T \Gamma'_{abcd} \eta' \]
\[ \psi' = *' \varphi' \]

Use \( \delta \psi = 0 \) to derive

\[ d \varphi' = \alpha \left[ \varphi' \right] \]
\[ d \psi' = \beta \left[ \varphi' \right] \]
\[ \psi' = *' \varphi' \]

\[ \alpha_{abcd} = 8A'_{[a} \varphi'_{bcd]} - 8B'_{[a} \psi'_{bcd]e} \]
\[ \beta_{abcd} = 10A'_{[a} \psi'_{bcd]} - 40B'_{[ab} \varphi'_{cd]} \]
Expanding to order $n$ in $\alpha'$

\[
\begin{align*}
    d\varphi' \big|_n &= \alpha [\varphi'] \big|_n = d \chi_n \\
    d\psi' \big|_n &= \beta [\varphi'] \big|_n = d \xi_n
\end{align*}
\]

contains the unknown which is the order $n$
of $\varphi'$

$\psi' = \ast \varphi'$

involves $\varphi'$ to order less than $n$

Exactness of $\alpha$ and $\beta$ is not only necessary but also sufficient....

There exists a solution of $\delta \psi = 0$ to all orders in $\alpha'$!

K. B., D. Robbins,
E. Witten, 1404.2460
II. Interpretation In 3d And 4d Field Theory
M-Theory On A $G_2$ Manifold

We wish to describe the fluctuations around the background...

$\mathbb{R}^{1,3} \times M$

compact $G_2$ manifold

... these include massless states as well as massive KK modes.
Guiding Principles

Assemble fields into 4d superfields

Keep locality along space-time and $M$.

$$\phi = \phi(x, y)$$

$$C = \frac{1}{3!} C_{abc}(x, y) dy^a \wedge dy^b \wedge dy^c$$

11d fields decompose into many 4d fields

$$C_{MNP}, G_{MN} \rightarrow \left\{ C_{abc}, C_{ab\mu}, C_{a\mu\nu}, C_{\mu\nu\rho}, g_{ab}, g_{a\mu}, g_{\mu\nu} \right\}$$

$$g_{\mu\nu} = g_{\mu\nu}(x, y)$$
Manifest Global 4d Supersymmetry

The coordinates of flat 4d superspace are \((x^\mu, \theta, \bar{\theta})\).

Superfields are functions of these coordinates...

**Chiral superfields**

\[
\bar{D}_{\dot{\alpha}} \Phi = 0
\]

\[
\Phi(x, \theta) = C(x_+) + \sqrt{2}\theta \psi(x_+) + \theta\theta F(x_+)
\]

\[
C(x) = \hat{\phi}_{abc}(x, y) + iC_{abc}(x, y)
\]

\[
x_\pm^\mu = x^\mu \pm i\bar{\theta}\sigma^\mu \theta
\]
Action For Chiral Superfields

\[ I = \frac{1}{2} \int d^4 x \left[ K \left( \Phi, \Phi^\dagger \right) \right] \bigg|_D + \int d^4 x \left[ f \left( \Phi \right) \right] \bigg|_F + c.c. \]

Lagrangian density for bosonic fields

\[ \mathcal{L} = - \int_{M \times M} d^7 y d^7 y' \frac{\delta^2 K}{\delta \mathcal{C}(y) \delta \mathcal{C}(y')} \left( \partial_\mu \mathcal{C}(y) \partial^\mu \mathcal{C}(y') - F(y) F(y') \right) + 2 \text{Re} \int_M d^7 y \frac{\delta f(C)}{\delta \mathcal{C}(y)} F(y) \]

Example:

\[ K = \int_M d^7 y \sqrt{g(x, y)} \]
A good candidate is

\[ f(\Phi) \sim \int_M \Phi \wedge d\Phi \]

In a supersymmetric ground state

\[ \delta f = 0 \Rightarrow d\Phi = 0 \Rightarrow d\hat{\phi} = 0, G_4 = 0 \]

Comparing with the previous results \( d\varphi' = \alpha = d\chi \)

\[ \hat{\phi} = \varphi' - \chi \]
Kähler Metric

$C_{abc}(y)$ are coordinates of an infinite dimensional Kähler manifold.

11d gauge transformations $\delta C = d\Lambda$ give rise to isometries

$$C \rightarrow C' = C + id\Lambda$$

The Kähler form is invariant

$$J \rightarrow J' = J + d\left(i_{d\Lambda}J\right) + i_{d\Lambda}(dJ)$$

Poincare Lemma:

$$i_{d\Lambda}J = dP_{d\Lambda}$$

$P_{d\Lambda}: V \rightarrow \mathbb{C}$

$\Lambda \in V$, the space of 2-forms mod closed 2-form
\[ J = \int d^7 y d^7 y' \frac{\delta^2 K}{\delta C_{abc}(y) \delta \tilde{C}_{def}(y')} \delta C_{abc}(y) \land \delta \tilde{C}_{def}(y') \]

\[ J = \int d^7 y d^7 y' \frac{\delta^2 K}{\delta \hat{\phi}_{abc}(y) \delta \hat{\phi}_{def}(y')} \delta C_{abc}(y) \land \delta \hat{\phi}_{def}(y') \]

\[ i_{d\Lambda} J = \int d^7 y d^7 y' \frac{\delta^2 K}{\delta \hat{\phi}_{abc}(y) \delta \hat{\phi}_{def}(y')} \partial\left[ a \Lambda_{bc}(y) \land \delta \hat{\phi}_{def}(y') \right] \]

\[ = \int d^7 y' \delta \hat{\phi}_{def}(y') \frac{\delta}{\delta \hat{\phi}_{def}(y')} \left[ \int d^7 y \nabla_a \left( \frac{\delta K}{\delta \hat{\phi}_{abc}(y)} \right) \Lambda_{bc}(y) \right] \]

\[ = dP_{d\Lambda} = d \langle \Lambda, \mu \rangle \]
Since $P_{d\Lambda} \in V^*$:

$$P_{d\Lambda} = \langle \Lambda, \mu \rangle$$

Inner product of differential forms

Space-time action includes a term

$$I \sim \int d^4x \langle D, \mu \rangle = \int d^4x \int d^7y D_{bc} \nabla_a \left( \frac{\delta K}{\delta \hat{\phi}_{abc}} \right)$$

$D_{ab}$ auxiliary field of the vector multiplet containing $C_{ab\mu}$

Integrating out $D_{ab}$ and unbroken supersymmetry

$$\mu = 0 \Rightarrow \nabla_a \left( \frac{\delta K}{\delta C_{abc}(y)} \right) = 0 \Rightarrow \text{Closed 5-form!}$$
Summary

So far we turned $\delta \psi_a = 0$ order by order in $\alpha'$ into a PDE for

$$\varphi'_{abc} = \eta'^T \Gamma'_{abc} \eta'$$

To $O(\alpha'^3)$

$$d \varphi' = \alpha[\varphi] = d \chi$$

$$d \star' \varphi' = \beta[\varphi] = d \xi$$

... and the PDE is solvable. At $O(\alpha'^n), n > 3$, use induction over $n$, and construct order $n$ solution as a small perturbation around order $n - 1$ solution. This is solvable if

$$d \varphi' = \alpha[\varphi']$$

$$d \star' \varphi' = \beta[\varphi']$$

Cohomology conditions arise naturally as F- and D-term conditions for supersymmetry...

K. B., D. Robbins, E. Witten, 1404.2460
III. Massive Kaluza-Klein Modes In Superspace
Bosonic Fields

Fields

\[ C_{MNP} \rightarrow C_{abc}, C_{ab\mu}, C_{a\mu\nu}, C_{\mu\nu\rho} \]

\[ G_{MN} = \begin{pmatrix}
    h_{\mu\nu} + g_{cd} A_{\mu}^{c} A_{\nu}^{d} & g_{bc} A_{\mu}^{c} \\
    g_{ac} A_{\nu}^{c} & g_{ab}
\end{pmatrix} \]

\( M, N = 0, \ldots, 10 \)

\( \mu, \nu = 0, \ldots, 3 \)

\( a, b = 4, \ldots, 10 \)

Symmetries:

\[ \begin{aligned}
    C & \rightarrow C + d\Lambda \\
    x^{M} & \rightarrow x^{M} - \xi^{M}
\end{aligned} \]

4d system is very complicated but known in detail...
4 Summary

As a summary we present a concrete example. The space-time effective action for eleven-dimensional supergravity compactified to four dimensions is

\[
S = -\frac{1}{8\kappa^2} \int d\nu h^{\alpha\beta} \left( \frac{1}{2} g^{ab} g_{cd} + g^{ac} g^{bd} \right) D_\alpha g_{ab} D_\beta g_{cd} \\
+ \frac{1}{2\kappa^2} \int d\nu \left( h^{\beta\mu} h^{\gamma[\rho} h_{\alpha]}_{\nu} - \frac{1}{2} h^{\alpha\mu} h^{\beta[\nu} h_{\gamma]\rho} \right) D_\alpha h_{\beta\gamma} D_\mu h_{\nu\rho} \\
+ \frac{1}{4\kappa^2} \int d\nu f \left[ g^{ab} h^{\alpha[\beta} h^{\mu]}_{\nu} \hat{\nabla}_a h_{\alpha\beta} \hat{\nabla}_b h_{\mu\nu} - h^{\alpha\beta} \left( \frac{1}{2} g^{ab} g_{cd} + g^{ac} g^{bd} \right) \hat{\nabla}_a h_{\alpha\beta} \hat{\nabla}_b g_{cd} \\
+ (g^{pt} g^{qu} g^{rs} - \frac{1}{2} g^{ps} g^{qt} g^{ru} + g^{pr} g^{qu} g^{st}) \hat{\nabla}_r g_{pq} \hat{\nabla}_u g_{st} \right] - \frac{1}{8\kappa^2} \int d\nu f^{-1} (\mathcal{F}_{\mu\nu}^a)^2 \\
- \frac{1}{24\kappa^2} \int d\nu \left[ (D_\mu C_{abc} - 3 \partial_{[a} C_{bc]\mu})^2 + 4 f \left( \hat{\nabla}_{[a} C_{bcd]} \right)^2 \right] \\
- \frac{1}{16\kappa^2} \int d\nu f^{-1} (\mathcal{F}_{\mu\nu ab} + \mathcal{F}_{\mu\nu}^c C_{abc})^2 - \frac{1}{24\kappa^2} \int d\nu \left[ f^{-2} (F_{\mu\nu\rho a})^2 + \frac{f^{-3}}{4} (F_{\mu\nu\rho\sigma})^2 \right] \\
- \frac{55}{2^8 3^3 \kappa^2} \int dx^{\mu\nu\rho\sigma} dy^{abcdefg} F_{[\mu\nu\rho a} F_{abcd C_{efg}]}.
\]

(4.1)
Let's consider a toy model. Given is a collection of bosonic fields in $d$ space-time dimensions

$$\phi^{I_0}_0, \phi^{I_1}_1, \ldots, \phi^{I_p}_p, \ldots$$

1) The set of fields obtained from the reduction of $C_{MNP}$ to $d = 4$ is of this type.

2) $\phi^{I_p}_{[p]}$ is a space-time $p$-form taking values in vector space $V_p$. In the example of the $11 \to 4$ reduction $V_p$ is the vector space of differential forms on $M$ (7d manifold) of some degree.

3) We take the space-time to be flat $h_{\mu\nu} = \eta_{\mu\nu}$ and $A^a_{\mu} = 0$. Including these fields is work in progress... (K. B., M. Becker, W. Linch, D. Robbins, to appear)
Subject the $p$-form fields to gauge transformations

$$
\delta \phi_{[p]}^{I_p} = d \Lambda_{[p-1]}^{I_p} + \left( q^{(p)} \right)_{J_{p+1}}^{I_p} \Lambda_{[p]}^{J_{p+1}}
$$

$p$-1 form taking values in $V_p$

$$q^{(p)} : V_{p+1} \rightarrow V_p$$

are linear maps.

$p$ form taking values in $V_{p+1}$

These gauge transformations are inspired in the $11 \rightarrow 4$ example but is more general...

$$
\delta C_{\mu ab} = \partial_\mu \Lambda_{ab} - 2 \partial_{[a} \Lambda_{b]\mu}
$$
Field Strength

Define the field strength

\[ F_{[p+1]}^{I_p} = d\phi_{[p]}^I - \left(q^{(p)}\right)_{J_{p+1}}^I \phi_{[p+1]}^J \]

... which is gauge invariant if

\[ q^{(p)} \circ q^{(p+1)} = 0 \]

It is then natural to interpret \( q \) as the boundary operator for a chain complex...

\[ V_\bullet = \{ \ldots \xleftarrow{q^{(p+1)}} V_{p+1} \xrightarrow{q^{(p)}} V_p \xrightarrow{q^{(p-1)}} \ldots \xrightarrow{q^{(0)}} V_0 \} \]

Again, the 11 → 4 example is of this form.
Some fields are pure gauge, some massless and some massive. Example, if $d = 4$

$$V_4 \xrightarrow{q^{(3)}} V_3 \xrightarrow{q^{(2)}} V_2 \xrightarrow{q^{(1)}} V_1 \xrightarrow{q^{(0)}} V_0$$

$$q^{(1)} \circ q^{(2)} = 0$$

$$\text{Im } q^{(2)} \subseteq \text{Ker } q^{(1)} \subseteq V_2$$

Each $V_p$ can be written as direct sum of vector subspaces. Example

$$V_2 \cong \text{Im } q^{(2)} \oplus \text{Ker } q^{(1)} / \text{Im } q^{(2)} \oplus V_2 / \text{Ker } q^{(1)}$$

- Pure gauge
- Massless physical states
- Physical states (in general massive)
4d Superfields

The components can be embedded into 4d superfields (prepotentials)

\[ \alpha^A = \frac{1}{2} \left( \Phi^A + \overline{\Phi}^A \right) \bigg|_{\theta = \overline{\theta} = 0} \]

0-form

\[ A^I_a = -\frac{1}{4} \left( \sigma^a \right)_{a\dot{a}} \left[ D^\alpha, \overline{D}^{\dot{\alpha}} \right] V^I \bigg|_{\theta = \overline{\theta} = 0} \]

1-form

\[ B^M_{ab} = -\frac{i}{2} \left( \left( \sigma_{ab} \right)_{\alpha} D^\alpha \right)_{\beta} \left( \overline{\sigma}_{ab} \right)_{\dot{\alpha}} \left( \overline{D}^{\dot{\alpha}} \right)_{\dot{\beta}} \bigg|_{\theta = \overline{\theta} = 0} \]

2-form

\[ C^S_{abc} = \frac{1}{8} \varepsilon_{abcd} \sigma^d_{\alpha\dot{\alpha}} \left[ D^\alpha, \overline{D}^{\dot{\alpha}} \right] X^S \bigg|_{\theta = \overline{\theta} = 0} \]

3-form

\[ D^X_{abcd} = \frac{i}{8} \varepsilon_{abcd} \left( D^2 \Gamma^X - \overline{D}^2 \overline{\Gamma}^X \right) \bigg|_{\theta = \overline{\theta} = 0} \]

4-form

\[ a, b, \ldots = 0, \ldots, 3 \]

\[ \text{chiral superfield} \]

\[ \text{real scalar superfield} \]

\[ \text{chiral spinor superfield} \]

\[ \text{real scalar superfield} \]
Gauge Transformations In Superspace

\[ \delta \Phi^A = +q_I^A \Lambda^I \]

\[ \delta V^I = \frac{1}{2i} \left( \Lambda^I - \bar{\Lambda}^I \right) + q_I^I U^M \]

\[ \delta \Sigma^M_{\alpha} = -\frac{1}{4} \bar{D}^2 D_{\alpha} U^M + q_M^M \gamma^S_{\alpha} \]

\[ \delta X^S = \frac{1}{2i} \left( D^\alpha \gamma^S_{\alpha} - \bar{D}_{\dot{\alpha}} \bar{\gamma}^{S\dot{\alpha}} \right) + q_S^S \Xi^X \]

\[ \delta \Gamma^X = -\frac{1}{4} \bar{D}^2 \Xi^X \]

chiral superfield
real scalar superfield
chiral spinor superfield
real scalar superfield
Field Strength

Define the gauge invariant field strength superfields

\[ E^Z = -q_A^Z \Phi^A \]

\[ F^A = \frac{1}{2i} \left( \Phi^A - \bar{\Phi}^A \right) - q_I^AV^I \]

\[ W^I_{\alpha} = -\frac{1}{4} \bar{D}^2 D_{\alpha} V^I - q_m^I \Sigma_{\alpha}^M \]

\[ H^M = \frac{1}{2i} \left( D_{\alpha} \Sigma_{\alpha}^M - \bar{D}_{\dot{\alpha}} \bar{\Sigma}_{\dot{\alpha}}^M \right) - q_s^M X^S \]

\[ G^S = -\frac{1}{4} \bar{D}^2 X^S - q_x^S \Gamma^X \]
Action

\[ S_{\text{kin}} = \frac{1}{2} \text{Re} \left( \int d^4 x \int d^2 \theta \tau_{IJ} W^I \alpha W^J \right) + \]
\[ \int d^4 x \int d^4 \theta \left( \frac{1}{2} g_{AB} F^A F^B - \frac{1}{2} g_{MN} H^M H^N + g_{ST} \bar{G}^S G^T \right) \]

real constants

This action is manifestly gauge invariant since the field strength superfields are gauge invariant and supersymmetric if we drop total derivatives....

Using the invariant field strengths actions can be obtained using real scalar combinations integrated over superspace or any chiral combination integrated over half of superspace ...
To make contact with the $11 \to 4$ example we need contributions to the Lagrangian density linear in prepotentials.

There exists a Chern-Simons superfield action linear in the prepotentials which is again supersymmetric after dropping total derivatives but is only gauge invariant when many terms are combined and when integrated over superspace…


Stay Tuned! More To Come...