Noncommutative Instantons and Reciprocity

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Based on

• MH&T.Nakatsu(Setsunan), (to appear)

Noncommutative Instantons and Reciprocity,
cf. (Proceedings:) ADHM construction of
noncommutative instantons [arXiv:1311.5227]
1. Introduction

- Non-Commutative (NC) spaces are defined by noncommutativity of spatial coordinates:

\[ [x^\mu, x^\nu] = i \theta^{\mu\nu} \quad \theta^{\mu\nu}: \text{NC parameter (real const.)} \]

(cf. CCR in QM : \[ [q, p] = i\hbar \])

(\( \rightarrow \) “space-space uncertainty relation” \( \rightarrow \))

\textbf{Resolution of singularity}

(\( \rightarrow \) new physical objects)

Ex) Resolution of small instanton singularity

(\( \rightarrow \) U(1) instantons) \quad [Nekrasov-Schwarz]
ASDYM eq. in 4-dim. with $G=U(N)$

- **ASDYM eq. (real rep.)**
  \[
  F_{12} = - F_{34}, \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu
  \]

- **Field strength**

- **Gauge field**
  \[
  A_\mu : (N \times N \text{ anti-Hermitian})
  \]

- There are two descriptions of **NC extension:**
  - Moyal-product formalism (deformation quantization)
  - Operator formalism (Connes’ theory)

\[
\begin{align*}
&\mu, \nu = 1, 2, 3, 4 \\
&F_{12} = - F_{34},
&F_{13} = - F_{42},
&F_{14} = - F_{23}.
\end{align*}
\]

\[
(\Leftrightarrow F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0, \quad F_{z_1z_2} = 0 \quad (\text{cpx. rep.}))
\]
NC ASDYM eq. with G=U(N) in Moyal

• NC ASDYM eq. (real rep.)

\[ F_{01}^* = - F_{23}^* , \quad (F_{\mu\nu}^* := \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \ast A_\nu - A_\nu \ast A_\mu) \]

\[ F_{02}^* = - F_{31}^* , \]

\[ F_{03}^* = - F_{12}^* \]

(Spell: All products are Moyal products.)

\[
\begin{align*}
\theta^{\mu\nu} &= \\
&= \begin{bmatrix}
0 & \theta^1 \\
-\theta^1 & 0 \\
O & 0 \\
O & -\theta^2 \\
\end{bmatrix}
\end{align*}
\]

Under the spell, we can calculate:

\[
f(x) \ast g(x) := f(x) \exp \left( \frac{i}{2} \theta^{\mu\nu} \tilde{\partial}_\mu \tilde{\partial}_\nu \right) g(x)
\]

\[
= f(x) \cdot g(x) + \frac{i}{2} \theta^{\mu\nu} \partial_\mu f(x) \partial_\nu g(x) + O(\theta^2)
\]

Under the spell, the solution is deformed:

\[
[x^\mu, x^\nu] := x^\mu \cdot x^\nu + \frac{i}{2} \theta^{\mu\nu} - (x^\nu \cdot x^\mu - \frac{i}{2} \theta^{\mu\nu})
\]

\[
A(x, \theta) = A^{(0)}(x) + \theta A^{(1)}(x) + \theta^2 A^{(2)}(x) + \cdots,
\]

\[= i \theta^{\mu\nu} : \text{NC Space!} \]
Plan of this talk

1. Introduction (10min.)
2. ADHM construction (15min.)
3. Proof of the (ADHM) reciprocity (10min.)
4. Conclusion and Discussion (5-10min.)
2. Atiyah-Drinfeld-Hitchin-Manin Construction based on reciprocity for the instanton moduli space

4dim. ASDYang-Mills eq.  
(Difficult)

\begin{align*}
F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} &= 0 \\
F_{z_1z_2} &= 0
\end{align*}

N × N PDE

Sol. = instantons  
(G = U(N), C_2 = k)

\[ A_\mu : N \times N \]

Gauge trf.:

\[ A_\mu \mapsto g^{-1}A_\mu g + g^{-1}\partial_\mu g, \quad g \in U(N) \]

ADHM eq. (≒0dim. ASDYM)  
(Easy)

\begin{align*}
[B_1, B_1^+] + [B_2, B_2^+] + I I^+ − J^+ J &= 0 \\
[B_1, B_2] + IJ &= 0
\end{align*}

k × k Matrix eqs.

Sol. = ADHM data  
(G = 'U(k)')

\[ B_{1,2} : k \times k, \quad I : k \times N, \quad J : N \times k \]

Gauge trf.:

\begin{align*}
B_{1,2} &\mapsto \tilde{g}^{-1}B_{1,2}\tilde{g}, \quad \tilde{g} \in U(k) \\
I &\mapsto \tilde{g}^{-1}I, \quad J \mapsto J\tilde{g}
\end{align*}
Fourier-Mukai-Nahm transformation

**Beautiful reciprocity between instanton moduli on 4-tori and instanton moduli on the dual tori**

4dim. ASD Yang-Mills eq. on a 4-torus:

\[
\begin{align*}
F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} &= 0 \\
F_{z_1\bar{z}_2} &= 0
\end{align*}
\]

N × N PDE

Sol. = instantons (G = U(N), C₂ = k)

\[A_\mu : N \times N\]

1:1 (reciprocity)

Define the maps F & G, & G ∘ F = id. & F ∘ G = id.

On a 4-torus

4dim. ASD Yang-Mills eq. on the dual torus:

\[
\begin{align*}
\tilde{F}_{\xi_1\bar{\xi}_1} + \tilde{F}_{\xi_2\bar{\xi}_2} &= 0 \\
\tilde{F}_{\xi_1\bar{\xi}_2} &= 0
\end{align*}
\]

k × k PDE

Sol. = the dual instantons (G = U(k), C₂ = N)

\[\tilde{A}_\mu : k \times k\]

On the dual 4-torus
Fourier-Mukai-Nahm transformation

Beautiful reciprocity between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASD Yang-Mills eq. on a 4-torus

\[ F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0 \]
\[ F_{z_1z_2} = 0 \]

N × N PDE

Sol. = instantons
(G = U(N), C_2 = k)

A_\mu (x) = \left\langle V, \partial_\mu V \right\rangle_\xi

4dim. ASD Yang-Mills eq. on the dual torus

\[ \tilde{F}_{\xi_1\bar{\xi}_1} + \tilde{F}_{\xi_2\bar{\xi}_2} = 0 \]
\[ \tilde{F}_{\xi_1\xi_2} = 0 \]

k × k PDE

Sol. = the dual instantons
(G = U(k), C_2 = N)

\[ \xi \mapsto F \] (Dirac eq.)

\[ \tilde{A}_\mu (\xi) : k \times k \]

\[ \nabla^+ V = \bar{e}^\mu \otimes (\frac{\partial}{\partial \xi^\mu} + \tilde{A}_\mu - i x^\mu) V = 0 \]
\[ \bar{e}^\mu := (i \sigma_a, 1_2) \]

V : 2k × N

Family index thm.

On a 4-torus : x_\mu

On the dual 4-torus : \xi_\mu
Fourier-Mukai-Nahm transformation

Beautiful reciprocity between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASD Yang-Mills eq. on a 4-torus

\[ F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0 \]
\[ F_{z_1\bar{z}_2} = 0 \]

N × N PDE

Sol. = instantons (G = U(N), C2 = k)

\[ A_\mu(x) : N \times N \]

Dirac eq. \[ G \]

4dim. ASD Yang-Mills eq. on the dual torus

\[ \widetilde{F}_{\xi_1\bar{\xi}_1} + \widetilde{F}_{\xi_2\bar{\xi}_2} = 0 \]
\[ \widetilde{F}_{\xi_1\xi_2} = 0 \]

k × k PDE

Sol. = the dual instantons (G = U(k), C2 = N)

\[ \widetilde{A}_\mu(\xi) = \left\langle \psi, \tilde{\partial}_\mu \psi \right\rangle_{\bar{\xi}} \]

k × k

\[ \psi : 2N \times k \]

Family index thm.

On a 4-torus : \( x_\mu \)

On the dual 4-torus : \( \xi_\mu \)

\[ \bar{e}_\mu D_\mu \psi = \bar{e}_\mu \otimes (\frac{\partial}{\partial x_\mu} + A_\mu - i\xi_\mu)\psi = 0 \]
Fourier-Mukai-Nahm trf. (radii of the torus $\to \infty$) 
reciprocity between instanton moduli on $\mathbb{R}^4$ 
and instanton moduli on "1pt." [cf. van Baal, hep-th/9512223]

4dim. ASD Yang-Mills eq.

$$F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0$$
$$F_{z_1 \bar{z}_2} = 0$$

$N \times N$ PDE

Sol. = instantons  
($G = U(N)$, $C_2 = k$)

$$A_\mu = V^+ \partial_\mu V$$

On a 4-torus $\to \mathbb{R}^4$

0dim. ASD Yang-Mills eq.

$$\tilde{F}_{\mu \nu} := \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu + [\tilde{A}_\mu, \tilde{A}_\nu]$$

$$\tilde{F}_{\xi_1 \bar{\xi}_1} + \tilde{F}_{\xi_2 \bar{\xi}_2} = 0$$

$$\tilde{F}_{\xi_1 \bar{\xi}_2} = 0$$

$N \times N$ PDE

Sol. = "dual instantons"  
($G = U(k)$, "$C_2 = N$"

map $F$ (0dim Dirac eq.)

$$\tilde{A}_\mu : k \times k$$

$V : 2k \times N$

Linear alg.

On the dual 4-torus $\to 1$ pt.
Atiyah-Drinfeld-Hitchin-Manin (ADHM) Construction based on the following reciprocity

4dim. ASDYang-Mills eq.

\[ F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2} = 0 \]
\[ F_{z_1 z_2} = 0 \]

Sol.=instantons
\((G=U(N), C_2 =k)\)
\[ A_\mu : N \times N \]

1:1
Proved in the same way as the Nahm trf.

ADHM eq. (\(\Leftrightarrow\) 0dim. ASDYM)
RHS is in fact \([z_1, \bar{z}_1] + [z_2, \bar{z}_2]\)

\[ [B_1, B_1^+] + [B_2, B_2^+] + I I^+ - J^+ J = 0 \]
\[ [B_1, B_2] + IJ = 0 \]
\(k \times k\) matrix eq.

Sol.=ADHM data
\((G='U(k)'\)\)
\[ B_{1,2} : k \times k, \]
\[ I : k \times N, \quad J : N \times k \]
ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction
Ex.) Commutative BPST instanton \((N=2, k=1)\)

4dim. ASDYang-Mills eq.

\[
F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0
\]

\[
F_{z_1\bar{z}_2} = 0
\]

\[N \times N\] PDE

BPST instanton
\((G=U(2), C_2 =1)\)

\[
A_\mu = \frac{i(x-b)^\nu \eta_{\mu\nu}^{ASD}}{(z-\alpha)^2 + \rho^2}, \quad 2 \times 2
\]

\[
F_{\mu\nu} = \frac{2i\rho^2}{((z-\alpha)^2 + \rho^2)^2} \eta_{\mu\nu}^{ASD}
\]

ADHM eq. \((\Rightarrow 0\text{dim. ASDYM})\)

\[
\mu_R = [B_1, B_1^+] + [B_2, B_2^+] + II^+ - J^+ J = 0
\]

\[
\mu_C = [B_1, B_2] + IJ = 0
\]

\[k \times k\] matrix eq.

Sol. = ADHM data
\((G=`U(1)`)\)

\[
B_{1,2} = \alpha_{1,2}, \quad 1 \times 1
\]

\[
I = (\rho, 0), \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}
\]

\[
\rho \to 0
\]

: singular

\[
\rho \to 0
\]

: singular

\[
\rho \to 0
\]

: singular
ADHM (Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) NC BPST instanton (N=2, k=1)

NC ASDY Yang-Mills eq.

\[ F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0 \]
\[ F_{z_1z_2} = 0 \]

N × N PDE

NC BPST instanton
(G=U(2), C=1)

By calculation of TrFA\!F
\[ A_\mu, F_{\mu\nu} \] : exact sol.

Do k × k ADHM data give Instanton number k in general? (We prove this.)

NC ADHM eq.

\[ \mu_R = [B_1, B_1^+] + [B_2, B_2^+] + II^+ - J^+ J = \zeta \]
\[ \mu_C = [B_1, B_2] + IJ = 0 \]

k × k matrix eq.

Sol.: ADHM data
(G=`U(1)’)

Resolution of the singularity!

\[ B_{1,2} = \alpha_{1,2}, \]

\[ I = (\sqrt{\rho^2 + \zeta}, 0), \quad J = \begin{pmatrix} 0 \\ \rho \end{pmatrix} \]

Fat by \( \zeta \)!

\[ \rho \rightarrow 0 \quad : \text{regular!} \]
3. Proof of the reciprocity: \((\text{inst}) \leftrightarrow (\text{ADHM})\)

**NC instanton** \[ G \] **NC ADHM** \[ F \]

\[ A_\mu : N \times N \]

\[ B_{1,2} : k \times k, \]

\[ I : k \times N, \quad J : N \times k \]

(i) ASD (ASDYM eq.)
(ii) \( C_2 = k \)
(iii) \( \nabla^2 \) has inverse

(i) ASD (ADHM eq.)
(ii) matrix size = \( k, N \)
(iii) \( \nabla^2 \) has inverse

(iii) is automatically satisfied in the noncommutative situation

[Maeda-Sako] [Nakajima] (For any \( \theta \) [MH, Nakatsu])

**Proof of the one-to-one** \( \Leftrightarrow \) **Define the maps** \( F \) & \( G \), & \( G \circ F = \text{id.} \) & \( F \circ G = \text{id.} \)
\( F: (\text{ADHM}) \rightarrow (\text{inst}): \text{ADHM construction} \)

\[
A_\mu = V^+ \ast \partial_\mu V : N \times N
\]

\[
\begin{pmatrix}
I^+ & J \\
\bar{z}_2 - B_2^+ & -(z_1 - B_1) \\
\bar{z}_1 - B_1^+ & z_2 - B_2
\end{pmatrix}
\]

\( B_{1,2} : k \times k, \quad I : k \times N, \quad J : N \times k \)

0 dim Dirac op. \((N + 2k) \times 2k\)

(i) ASD (ASDYM eq.) \[\text{[Nekasov-Schwarz]}\]

(ii) \( C_2 = k \) \[\text{[MH Nakatsu],...}\]

We prove the NC version of the formula: \[\text{cf.}[\text{Atiyah, Hori}]\]

\[
\int d^4x \text{Tr}_N F^*_{\mu\nu} \ast F^*_{\mu\nu} = -\int d^4x \text{Tr}_{2k} \Omega^*_{\mu\nu} \ast \Omega^*_{\mu\nu}
\]

where \( \Omega_{\mu\nu} := \partial_\mu \omega - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu] \)

\[
\omega_\mu := \tilde{\nabla}^+ \ast \partial_\mu \tilde{\nabla}, \quad \tilde{\nabla} := \nabla \ast (\nabla^+ \ast \nabla)^{1/2}
\]

\( 2k \times 2k, \quad (N + 2k) \times 2k \)

\[
\text{cf. } \text{ch}(E \oplus F) = \text{ch}(E) + \text{ch}(F)
\]
F : (ADHM) \rightarrow \text{(inst)} : \text{ADHM construction}

**NC instanton**

\[ V^+ \ast V = 0, \quad V^+ \ast V = 1_N \]

\[ A_\mu = V^+ \ast \partial_\mu V : N \times N \]

\[ \nabla = \begin{pmatrix} I^+ & J \\ \bar{z}_2 - B_2^+ & -(z_1 - B_1) \\ \bar{z}_1 - B_1^+ & z_2 - B_2 \end{pmatrix} \]

\[ B_{1,2} : k \times k, \quad I : k \times N, \quad J : N \times k \]

0 dim \text{Dirac op.} \cdot (N + 2k) \times 2k

**NC ADHM**

Then:

\[ C_2 := \frac{1}{16\pi^2} \int d^4x \text{Tr}_N F_{\mu\nu}^* \ast F_{\mu\nu}^* = -\frac{1}{16\pi^2} \int d^4x \text{Tr}_{2k} \Omega_{\mu\nu}^* \ast \Omega_{\mu\nu}^* \]

\[ = \frac{1}{24\pi^2} \int_{S^3} \text{Tr}_k 1_k \cdot \text{Tr}_2 (g^{-1}dg)^3 = k, \quad \text{g} := \frac{x^\mu e_\mu}{r} \]

comes from the size of the ADHM data!
G : (inst)→(ADHM): inverse construction

NC instanton \[ \bar{e}_\mu D_\mu \ast \psi = 0, \quad \int d^4 x \psi^+ \ast \psi = 1_k \]

NC ADHM

\[ A_\mu : N \times N \quad e^\mu D_\mu : 4 \text{dim Dirac op.} \]

\[ B_{1,2} = \int d^4 x z_{1,2} \ast \psi^+ \ast \psi : k \times k, \]

\[ \psi \approx \frac{I^+, J}{r^3} : N \times k \]

(i) ASD (ASDYM eq.) (i) ASD (ADHM eq.)
(ii) C_2=k (ii) matrix size= k, N

[Maeda-Sako2009] proves existence of the Dirac zero-mode by a formal power expansion of \( \theta \), recursively.

\[ G \circ \text{id} : (\text{ADHM}) \rightarrow (\text{inst}) \rightarrow (\text{ADHM}) \]

NC ADHM $\rightarrow$ NC instanton $\rightarrow$ NC ADHM

$B_{1,2}, I, J$

$\psi = \psi(B, I, J, V)$

$\psi = V^+ \ast C_f$

Find $B'_{1,2} = B_{1,2}$?

$I' = I$? $J' = J$?

The answer

$$\bar{e}_\mu D_\mu \ast \psi = 0, \quad \int d^4 x \psi^+ \ast \psi = 1_k$$

$$B'_{1,2} = \int d^4 x \ z_{1,2} \ast \psi^+ \ast \psi = \cdots = B_{1,2}$$

$$\psi \approx \frac{I''^+, J'}{r^3} = \cdots = \frac{I^+, J}{r^3}$$

are shown

[Maeda-Sako]

[MH-Nakatsu]
\(F \circ G = \text{id} : \text{(inst)} \rightarrow \text{(ADHM)} \rightarrow \text{(inst)}\)

NC instanton \(\rightarrow\) 4dim. Dirac eq. \(\psi\) \(\rightarrow\) NC ADHM \(\rightarrow\) 0dim. Dirac eq.

\[ V = V(A_\mu, \psi) \]

\[ D^2 * V = -4\psi^+ C \]

\[ \nabla^+ * V = 0, \quad V^+ * V = 1_N \]

\[ A_\mu' = V^+ * \partial_\mu V = \cdots = A_\mu \]

[Maeda-Sako] assume the existence of \(V\).
[MH-Nakatsu] prove it

\(A_\mu' = A_\mu\) ?

in terms of the original instanton \(A_\mu\)
find

the answer

are shown (some existence proofs is also made by us)
Main result: We prove the ADHM reciprocity in the formal power series of θ-expansion for arbitrary noncommutativity (including ζ=0).

(i) ASD (ASDYM eq.)
(ii) \( C_2 = k \)

(i) ASD (ADHM eq.)
(ii) matrix size = \( k, N \)

• This is valid only in the region that the θ-expansions converge.

• We proceed to reveal the reciprocity in operator formalism. (mostly completed [MH-Nakatsu] )
G=U(N) NC ASDYM in operator formalism

- Take coordinates as operators (in 2dim):

\[
[\hat{x}, \hat{y}] = i\theta \xrightarrow{\text{complex}} [\hat{z}, \hat{\bar{z}}] = 2\theta \xrightarrow{\text{rescale}} [\hat{\alpha}, \hat{\alpha}^+] = 1
\]

field (infinite matrix):

\[
\hat{F}(\hat{z}, \hat{\bar{z}}) = \sum_{m,n}^\infty F_{mn} |m\rangle \langle n|
\]

integration

\[
2\pi\theta T \text{Tr}_\mathcal{H} \hat{F}(\hat{z}, \hat{\bar{z}})
\]

- NC ASDYM eq. (real rep.)

\[
\begin{align*}
\hat{F}^{01} &= -\hat{F}^{23}, \\
\hat{F}^{02} &= -\hat{F}^{31}, \\
\hat{F}^{03} &= -\hat{F}^{12}
\end{align*}
\]

\[
\theta^{\mu\nu} = \begin{bmatrix} 0 & -\theta^1 & O \\ \theta^1 & 0 & O \\ O & 0 & -\theta^2 \end{bmatrix} 
\]

\[
\hat{F} = \sum_{m_1,m_2,n_1,n_2}^\infty F_{m_1,m_2,n_1,n_2} |m_1,m_2\rangle \langle n_1,n_2|
\]

\[
H = \bigoplus C |n\rangle \quad n = 0,1,2,...
\]

Occupation number basis

\[
\begin{align*}
\text{ann op. cre op.} \\
\text{acting on Fock space:}
\end{align*}
\]

\[
\hat{F} = \begin{bmatrix} 0 & -\theta^1 \\ \theta^1 & 0 \\ O & 0 \end{bmatrix} \quad \Rightarrow \quad H_1
\]

\[
\begin{bmatrix} O & \theta^2 \\ \theta^2 & 0 \end{bmatrix} \quad \Rightarrow \quad H_2
\]
NC ADHM reciprocity in operator formalism

- **Strategy** is almost the same as the Moyal.
- **Set ups**: the Fock representation with a regularization:
  \[
  \text{Tr}_H \hat{F}(\hat{z}, \hat{\bar{z}}) = \lim_{L \to \infty} \sum_{n_1 + n_2 \leq L} \langle n_1, n_2 | \hat{F} | n_1, n_2 \rangle
  \]

**asymptotics**:

\[ e.g. \hat{A}_\mu - \hat{g}^{-1} \partial_\mu \hat{g} = O(L^{-3/2}) \quad (\forall \nu \approx r^{2\nu}) \]

- **There are several non-trivial points**;

Surface integration

\[
\sum_{n_1 + n_2 \leq L} (\partial^2 \hat{F})_{n_1, n_2; m_1, m_2} = \frac{8}{\zeta} \left\{ (L+1) \sum_{n_1, n_2 = L+1} \hat{F}_{n_1, n_2; m_1, m_2} - (L+2) \sum_{m_1, n_2 = L} \hat{F}_{n_1, m_2; m_1, n_2} \right\}
\]

- **Index theorem**... → We apply this to the NC COGT formula on instanton number
4. Conclusion and Discussion

- We discuss global solutions (instantons) of the Anti-Self-Dual Yang-Mills (ASDYM) eqs in the framework of the ADHM.
- NC extension $\rightarrow$ resolution of singularity
- We can construct local solutions in terms of quasideterminants in the framework of twistor
- NC extension $\rightarrow$ much easier (essential?)
- NC extension is interesting (natural and essential and perhaps necessary).
Local solution of NC ASDYM

\[ J_{[n]} = \begin{pmatrix} f_{[n]} - g_{[n]} b_{[n]}^{-1} e_{[n]} & -g_{[n]} b_{[n]}^{-1} \\ b_{[n]}^{-1} e_{[n]} & b_{[n]}^{-1} \end{pmatrix} \]

\begin{bmatrix}
0 & -1 & 0 & \cdots & 0 & 0 \\
1 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-n} & \Delta_{-n} \\
0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-n} & \Delta_{1-n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 & \Delta_{-1} \\
0 & \Delta_n & \Delta_{n-1} & \cdots & \Delta_1 & \Delta_0 \\
\end{bmatrix}

: compact formula in terms of quasideterminant!
(by Gelfand-Retakh)

- Proofs are much easier than commutative one!
- Quasi-determinants would give more essential formulation of (NC) soliton theory including KP.