Riemann-Hilbert problem for period integrals

Bong Lian
Brandeis University

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The big picture

Period integrals

- MIRROR
  SYMMETRY

- ALGEBRAIC
  GEOMETRY

- SPECIAL
  FUNCTIONS
Collaborators

Based on recent joint works with S. Bloch, A. Huang, V. Srinivas, S.-T. Yau and X. Zhu, and ongoing work with L. Fu and M. Zhu. Joint work with S. Hosono during 1994–2004 also provided much of the motivation for the current work.
Outline

- I. Set-up for the Riemann-Hilbert problem
- II. Recent developments
- III. Results and applications
  - Completeness question
  - Geometric description of solutions to GKZ systems
  - Large complex structure limits
  - Hyperplane conjecture
  - Topological proof of the GKZ solution rank formula
- IV. Ideas of proofs
1. Geometric set-up

Consider a family $\pi : \mathcal{Y} \to B$ of $d$-dimensional compact complex manifolds, with $Y_b := \pi^{-1}(b)$.
2. Geometric set-up

- The $H^k(Y_b)$ form a local system $E$ with the Gauss-Manin connection $\nabla$:

$$E := R^k\pi_*\mathbb{C}.$$  

We will consider the case $k = d = \dim Y_b$.

- The dual bundle $E^\vee$ has fibers $H_d(Y_b)$.

- Pairing induced by Poincaré $H_d(Y_b) \times H^d(Y_b) \hookrightarrow \mathbb{C}$:

$$\langle \cdot, \cdot \rangle: \mathcal{O}(E^\vee) \otimes \mathcal{O}(E) \to \mathcal{O}_B.$$  

- Local parallel sections of $E^\vee$:

$$\ker \nabla \simeq H_d(Y_b).$$
3. Period sheaf

- Fix a global section \( \omega \in \Gamma(B, E) \).

- **Definition:** The period sheaf \( \Pi(\omega) \) (with respect to \( \omega \)) of \( \mathcal{Y} \) is the sheaf of analytic functions on \( B \) generated by the germs

\[
\langle \gamma, \omega \rangle_b = \left( \int_{\gamma} \omega \right)_b \in \mathcal{O}_{B, b}, \quad \gamma \in \ker \nabla.
\]

These germs are called **period integrals** of \( \mathcal{Y} \).
4. Riemann-Hilbert problem for period integrals

**Problem:** Construct an explicit *complete* linear PDE system $\tau$ for the sheaf $\Pi \equiv \Pi(\omega)$. That is a system $\tau$ such that

$$\text{sol}(\tau) = \Pi.$$

**Goals & applications:**

- To compute explicitly periods $\int_\gamma \omega$ as power series
- To understand local monodromy of periods
- To count curves in algebraic varieties by (homological) mirror symmetry
- To construct interesting new classes of special functions
5. A toy example

▶ **Example:** \( X = \mathbb{P}^1 \). The family of points in \( X \) defined by the polynomial equations:

\[
 f(a, x) := a_0 x_0 x_1 + a_1 x_1^2 + a_2 x_2^2 = 0.
\]

For \( b \in B := \mathbb{C}^3 - \{4a_1a_2 - a_0^2 = 0\} \), \( Y_b \equiv \{f(b, x) = 0\} \) is a pair of points in \( \mathbb{P}^1 \).

▶ \( \mathbb{P}GL_2 \cong \Gamma(X, -K_X) \equiv \sum_i \mathbb{C}a_i \) induces 1st order differential operators of the form

\[
 Z_\xi := \sum_{ij} \xi_{ij} a_i \frac{\partial}{\partial a_j}, \quad (\xi \in \mathfrak{sl}_2).
\]

▶ Solution to the RH problem for \( \Pi \) in this case is the differential system \( \tau \):

\[
 (\frac{\partial}{\partial a_0})^2 - \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2}, \quad \sum_i a_i \frac{\partial}{\partial a_i} + 1, \quad Z_\xi \quad (\xi \in \mathfrak{sl}_2).
\]
6. Classical history: Riemann-Hilbert problem

- Inspired by Euler and Gauss, who studied elliptic integrals

\[
\int_{\gamma} \frac{dx}{\sqrt{x(x-1)(x-\lambda)}},
\]

Riemann considered the problem of constructing an ordinary differential equation with a prescribed monodromy.

- Hilbert formulated a generalization of Riemann’s problem in his famous 1900 Paris ICM address, as his 21st problem.

- A generalized version of the problem known as the RH correspondence was proved for algebraic connections with regular singularities by [Deligne ’70], and more generally for regular holonomic D-modules independently by [Kashiwara ’80-’84] and [Mebkhout ’80-’84].

The Riemann-Hilbert problem for periods asks for an explicit PDE system for periods of algebraic varieties. The abstract RH correspondence alone does not provide an explicit construction or description, hence it is not enough for our applications.
7. Returning to period integrals …

One set-up that was the focus of a lot of recent work is the case of hypersurfaces $Y$ in some fixed ambient smooth projective variety $X$.

- We will fix a very ample line bundle $L$ on $X$, and consider the universal family $\mathcal{Y}$ defined over $B := \{ \text{all smooth hyperplane sections of } L \}$:
  $$\mathcal{Y} := \{(b, x) \in B \times X | b(x) = 0\} \xrightarrow{\pi} B.$$

- Usual trichotomy of cases:
  
  (1) $K_Y > 0$ (general type)
  
  (2) $K_Y = 0$ (Calabi-Yau)
  
  (3) $K_Y < 0$ (Fano).

We will mostly focus on the CY case. Results discussed will carry over to the other two cases with some modifications.

- In this case, [L-Yau ’12] showed that the flat bundle $E$ associated to $\mathcal{Y}$ always admits a canonical section $\omega$ (details later). We now proceed to understand the RH problem for the period sheaf $\mathcal{P}(\omega)$. 
8. Two approaches to the RH problem

- **Classical reduction-of-pole method:** [Dwork-Griffiths] Use integration by parts to compute linear relations among partial derivatives

\[
\partial^\alpha \left( \int_\gamma \omega \right)_b
\]

and carry out elimination of variables.

- This method works in many examples, but not effective in general, especially when \( \dim B \) is large.

- **The symmetry-constraints approach:** First, use symmetry of \( X \), and geometric constraints of the family \( \mathcal{Y} \) to construct a candidate PDE system \( \tau \), such that à priori

\[
sol(\tau) \supset \Pi.
\]

Then, decide if \( \tau \) is complete, i.e. if ‘\( \subset \)’ also holds. Otherwise, describe the obstruction.
9. Completeness Problem

- When $X$ is a toric manifold, [Batyrev '93] [Hosono et al '94-'95] showed that periods of the universal CY family in $X$ are solutions to a GKZ system. It was known that this GKZ system is never complete.

- Many authors were able to modify this system to construct one that is complete in special cases. Hosono et al and others constructed additional differential operators for the periods, extending the GKZ system.

- It was conjectured that the eGKZ is complete when $X$ is a product of $\mathbb{P}^n$, hence it would solve the RH problem in this case.

- Unfortunately, the conjecture was not decided even for the quintics family in $X = \mathbb{P}^4$!
10. Completeness Problem

▶ Generalization: For arbitrary $X$, [L-Yau ’12] constructed a new class of PDE systems we call tautological systems, and showed that periods of $\mathcal{Y}$ are solutions to such a system $\tau$. These systems generalize previous constructions for $K_Y = 0$, and they also include the cases of $K_Y < 0$ and $K_Y > 0$.

▶ For a given $X$, the tautological system $\tau$ in the CY case can be essentially characterized by a choice of symmetry group $G \subset \text{Aut} X$, together with the geometric constraints on the family $\mathcal{Y}$ provided by $L = -K_X$ (details later).

▶ Completeness Problem: For $G = \text{Aut} X$, when is the system $\tau$ complete, i.e.

$$\text{sol}(\tau) = \Pi(\omega)$$
11. Semi-periods

- Consider toric $n$-fold $X$ with symmetry group the torus $T$. In this case, the tautological system $\tau$ reduces to a standard GKZ system $\tau_{GKZ}$, which is incomplete.

- [Avram-Derrick-Jancic ’95] studied the extra solutions of $\tau_{GKZ}$ and called them semi-periods. They conjectured that semi-periods are integrals over certain singular chains with suitable boundary conditions.

- [Candelas-de la Ossa-Rodriguez Villegas ’00] showed that semi-periods are essential for ingredients for counting points of the CY variety $Y_b$ over finite fields.

**Question:** Can we give a precise geometrical description of the solutions to $\tau_{GKZ}$: construct the conjectural singular chains of Avram et al and describe their boundary conditions?
12. Large Complex Structure Limits

- **LCS Limits:** Construct a singular degeneration \( Y_\infty \), called an LCS limit, in the family \( \mathcal{Y} \) such that any nearby fiber \( Y_b \) contains a unique (local) monodromy invariant cycle in \( H_{n-1}(Y_b)_{\text{van}} \), with nonzero period integral.

- The unique monodromy invariant cycle is expected to be the torus cycle in the SYZ fibration of \( Y_b \).

- The monodromy invariant cycle condition is almost equivalent to the topological condition

\[
\dim H_n(X - Y_\infty) = 1.
\]

We call such a degenerate \( Y_\infty \) a rank 1 fiber of the family \( \mathcal{Y} \).

- Generalizing [Candelas-de la Ossa-Greene-Parkes ’91] and many other constructions, [Hosono-L-Yau ’97] show that for \( X \) a toric variety, then \( D := \) the union of \( T \)-invariant divisor in \( X \) is an LCS limit of the CY family \( \mathcal{Y} \) in \( X \).

**Example:** \( X = \mathbb{P}^n \), \( Y_\infty = D = \{x_0x_1 \cdots x_n = 0\} \) is a rank 1 fiber (and in fact an LCS limit) since

\[
H_n(X - Y_\infty) = H_n((S^1)^n).
\]

has dimension 1.
13. **Hyperplane Conjecture**

- For a pair of Batyrev’s mirror toric manifolds $X, X^*$, [Hosono-L-Yau ’95] gave an explicit combinatorial formula for the solution of $\tau_{GKZ}$: consider the cohomology-valued function $B_X : \mathcal{U}_\infty \to H^\bullet(X^*)$

$$B_X(a) := \sum_{d \in M_+(X^*)} \prod_{k=1}^{d \cdot D^*} (D^* + k) \ O_{X^*}(d) \times a_0^{-d \cdot D^* - D^* - 1} \prod_i a_i^{d \cdot D_i^* + D_i^*}.$$

This function was constructed by ‘deregularizing’ the Gamma series solutions of [GKZ ’90].

- Here $M_+(X^*)$ is the set of lattice points in the Mori cone of $X^*$, the $D_i^*$ are the $T$-divisors, and $D^*$ the anticanonical divisor of $X^*$; $O_{X^*}(d)$ are certain combinatorially defined cohomology class of $X^*$, and $a_i$ are linear coordinates on the section space $\Gamma(X, -K_X)$ diagonalizing the torus action.

- **Theorem:** [HLY ’95] The solution sheaf $\text{sol}(\tau_{GKZ})$ is generated by $B_X(a)$ near the LCS limit $D$ for the CY family $\mathcal{Y}$ in $X$.

- **Hyperplane Conjecture:** [HLY ’95] The periods of $\mathcal{Y}$ are given by the cup product $B_X(a) \cup D^*$.

- M. Zhu gave a talk yesterday on recent progress on this conjecture.
14. Main results

On the completeness question:

- **Theorem:** [Bloch-Huang-L-Srinivas-Yau '13] The tautological system $\tau (\mathbb{P}^n, \mathbb{P}GL_{n+1})$ for the universal CY family $\mathcal{Y}$ is complete.

- **Theorem A (Completeness):** [Huang-L-Zhu '14] For any homogeneous $G$-variety $X^n$, $\tau (X, G)$ is complete iff $PH^n (X) = 0$.

- **Example:** For $X = \mathbb{P}^n$ or any $X$ of odd dimension, this shows that $\tau$ solves the RH problem.

- **Example:** $X = \mathbb{P}^k \times \mathbb{P}^k$ ($n = 2k$). Then $PH^n (X) \neq 0$, hence $\tau$ is never complete in this case.

- **Remark:** For $PH^n (X) \neq 0$, we will give a precise geometrical description of all solutions to $\tau$, and in particular, how exactly $\tau$ fails to be complete.
15. Main Results

On the semi-period question:

- **Theorem B (Chain Integral Isomorphism):** [Huang-L-Yau-Zhu '15]
  For any Fano toric manifold $X$, we have a canonical isomorphism
  \[
  H_n(X - Y_b, D - Y_b) \sim \text{sol}(\tau_{GKZ})_b, \quad C \mapsto \left( \int_C \frac{\Omega}{f} \right)_b.
  \]
  for an arbitrary section $b$. Here $\frac{\Omega}{f(b)}$ is a canonical rational form on $X$ with simple pole along $Y_b$.

- In other words, every analytic solution to $\tau_{GKZ}$ is the integral over a singular chain with boundary on the $T$-divisor $D$.

- This gives a precise topological description of the semi-periods of [Avram et al '95] and [Candelas et al '00].

- Moreover, the periods of the universal CY family correspond precisely to singular chains, which are given by tubes over cycles $\gamma$ in the generic fiber $Y_b$. 

\[
\text{Normal bundle of } Y_b \text{ in } X
\]

\[
tube C_\gamma
\]

\[
\gamma
\]
16. Main Results

On LCS limits :

- **Theorem:** [Bloch-H-Lian-Srinivas-Yau ’13] For \( X = G(k, N) \), the anticanonical divisor \( D = \{ x_1 \cdots k x_2 \cdots (k+1) \cdots x_N \cdots (k-1) = 0 \} \) is a rank 1 fiber of the universal CY family in \( X \).

- **Theorem C:** [Huang-L-Zhu ’14] For any homogeneous \( G \)-variety \( X^n \), the universal CY family in \( X \) admits a rank 1 fiber \( D \).

- **Sketch of construction:** Let \( B \) be a Borel subgroup of \( G \). Then the flag variety \( G/B \) has a canonical stratification [Reitsch ’05, Knutson-Lam-Speyer ’10]:

\[
G/B = \bigsqcup_{(u,w)} \left[ (B^- uB/B) \cap (BwB/B) \right].
\]

The projection \( G/B \to X \) yields a stratification on \( X \) (called the projected Richardson stratification.)

- Put \( D := \text{closure of the codimension 1 stratum in } X \). Lam observed that the Kazhdan-Lusztig conjecture implies

\[
\dim H_n(X - D) = 1.
\]

- Moreover, the divisor \( D \) is anticanonical, hence it is a rank 1 fiber of the universal CY family in \( X \).

- A result of Kostant-Luna implies the point \( D \) is *semistable*. Therefore \( D \) survives as a point in the moduli space of CYs in \( X \).
17. Further application

As another application, we give a purely topological proof of a famous formula of Gel’fand-Kapranov-Zelevinsky for the rank of solution space of $\tau_{GKZ}$.

- **Theorem:** [GKZ ’90]

  \[
  \text{generic solution rank of } \tau_{GKZ} = \text{vol}(P_L)
  \]

  where the right side is the volume of the Newton polytope of the space of sections of the anticanonical line bundle $L$.

- **Sketch of topological proof:** [Fu-Huang-L-Yau’16] Recall that

  \[
  \chi(X - Y_b, D - Y_b) = \chi(X) - \chi(Y_b) - \chi(D) + \chi(Y_b \cap D).
  \]

  By a Chern class calculation and the Riemann-Roch theorem, the right side $= (-1)^n \text{vol}(P_L)$. Then a vanishing cohomology argument shows that the left side is $(-1)^n \dim H_n(X - Y_b, D - Y_b)$. Finally, the *Chain Integral Isomorphism (Theorem B)* implies that this term is precisely

  \[
  = (-1)^n \times \text{generic solution rank of } \tau_{GKZ}.
  \]
18. Set-up

- **Data:**
  - $G$: complex algebraic group
  - $X$: a smooth projective $G$-variety

- **Notations:**
  - $L = -K_X$
  - $V^\vee := \Gamma(X, L)$
  - $f : V^\vee \times X \to L$, $(b, x) \mapsto b(x)$ universal section of $L$
  - $\pi : \mathcal{Y} \to B$ the universal CY family in $X$
  - $E = R^{n-1} \pi_* \mathbb{C}$ local system associated to $\mathcal{Y}$
  - $U = V^\vee \times X \setminus \{ f = 0 \}$ universal complement

- Recall the Weyl algebra $D_{V^\vee} :=$ the algebra of polynomial differential operators on $V^\vee$. Then

  $$D_{V^\vee} = \mathbb{C}[a_0, \ldots, a_m, \frac{\partial}{\partial a_0}, \ldots, \frac{\partial}{\partial a_m}]$$

  where $(a_0, \ldots, a_m)$ are linear coordinates on $V^\vee$. 
19. Set-up

- **Theorem:** [L-Yau ’12] There is a canonical section of $E$ of the form

$$\omega = \text{Res} \frac{\Omega}{f}$$

where $\Omega$ is a certain $G$-invariant $n$-form on a $\mathbb{C}^*$-principal bundle on $X$, and $\text{Res}$ is the Leray-Poincaré residue map.

- By the residue formula, periods of $Y$ (with respect to $\omega$) are:

$$\int_{\gamma} \omega = \int_{C_{\gamma}} \frac{\Omega}{f}.$$

![Diagram](attachment:diagram.png)
20. **Tautological systems**

Next, we discuss main ideas of proofs for Theorems A-C.

**Observations:**

1. $f^{-1}$ is a section defined on $U$, and $\mathcal{O}_U f^{-1}$ is a (cyclic) $D_{V^\vee}$-module generated by $f^{-1}$.

2. If $P \in D_{V^\vee}$ annihilating $f^{-1}$, then $P$ annihilates the periods:

$$P \cdot \int_{C_\gamma} \frac{\Omega}{f} = 0 \quad \text{(Geometric constraints)}.$$  

Let $J_X \subset D_{V^\vee}$ be the annihilating ideal of $f^{-1}$.

3. Since $G \varpi X$, for any $\xi \in g \equiv \text{Lie}(G)$, the vector field $Z_\xi$ induced on $V^\vee$ also annihilates the periods:

$$Z_\xi \cdot \int_{C_\gamma} \frac{\Omega}{f} = 0 \quad \text{(Symmetry)}.$$  

Let $Z_g$ be the space of vector fields induced on $V^\vee$ by $G \varpi X$.

**Definition:** Let $\tau \equiv \tau(X, G)$ be the cyclic $D_{V^\vee}$

$$\tau = D_{V^\vee} / (J_X + D_{V^\vee} Z_g).$$

This $D_{V^\vee}$-module is called a **tautological system**.

By observations 1-3, it follows immediately that every period integral for the family $\mathcal{Y}$ is an analytic solution to $\tau$. 

21. Explicit algebraic description of $\tau$

- Since $L$ is very ample, we have a canonical ‘tautological’ embedding
  
  \[ X \hookrightarrow \mathbb{P}V \equiv \mathbb{P}^m \]

  where we identify $\mathbb{C}[V] \equiv \mathbb{C}[a_0^\vee, \ldots, a_m^\vee]$. Let $I_X \subset \mathbb{C}[V]$ be the vanishing ideal of $X \subset \mathbb{P}^m$.

- Define the Fourier transform of $I_X$:

  \[ \tilde{I}_X = \{ P(\frac{\partial}{\partial a_0}, \ldots, \frac{\partial}{\partial a_m}) \mid P(a_0^\vee, \ldots, a_m^\vee) \in I_X \} \subset \mathbb{C}[\frac{\partial}{\partial a}]. \]

- **Theorem**: (Annihilator of $f^{-1}$)

  \[ J_X = D_{V^\vee} \tilde{I}_X + D_{V^\vee} \left( \sum a_i \frac{\partial}{\partial a_i} + 1 \right). \]
22. Basic properties and examples of $\tau$

- $\tau$ makes sense if we replace $I_X$ by any $G$-invariant ideal $I \subset \mathbb{C}[V]$, and twist the vector fields $Z_g$ and the Euler vector field by a Lie algebra character. The more general set-up is needed to deal with families of Fano and general type varieties.

- $\tau$ is a **regular holonomic** $D$-module if $X$ has only finite number of $G$ orbits. That is to say, any formal power series solution is analytic; the sheaf of analytic solutions has finite rank at each point.

- $\tau$ has **solution rank** which is bounded above by the degree of the tautological embedding $X \hookrightarrow \mathbb{P}V$ if the ring $\mathbb{C}[V]/I_X$ is Cohen-Macaulay.

- **Example:** $X = \mathbb{P}^n$. The annihilator ideal $J_X$ is generated by the differential operators:

\[
\frac{\partial}{\partial a_\mu} \frac{\partial}{\partial a_\nu} - \frac{\partial}{\partial a_\mu'} \frac{\partial}{\partial a_\nu'} \quad (\mu + \nu = \mu' + \nu')
\]

\[
\sum_\mu a_\mu \frac{\partial}{\partial a_\mu} + 1
\]
23. Special cases of $\tau$

- $\tau(X, G)$ specializes to a **GKZ system** [GKZ '90] if $X$ is a toric variety and $G = T$ is the dense torus.

- $\tau(X, G)$ specializes to a **Kapranov system** [Kapranov '97] if $X$ a certain *wonderful compactification* of a semisimple group $G$.

- $\tau(X, G)$ has the following explicit description if $X$ is $G$-homogeneous: its ‘constraints’ operators are precisely given by

$$
(C_g - \lambda) \sum_{ij} C \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_j}
$$

$C_g =$ quadratic Casimir of $g; \lambda$ is certain distinguished eigenvalue of $C_g \in \text{End } V^\vee$. This follows from an old result of Kostant.
24. **Cohomological Description of $\tau$**

- Assume the *finite-orbit condition* on the $G$-variety $X$.

- Put
  
  $$D_{X,g} := D_X \otimes_g \mathbb{C},$$
  
  twisted D-module on $X$

- $\pi^\vee : U \rightarrow V^\vee$
  
  affine projection

- $\pi_+^\vee : D^b_h(U) \rightarrow D^b_h(V^\vee)$
  
  the derived pushforward

- $M \mapsto R\pi_+^\vee(\Omega^\bullet_{U/V^\vee} \otimes M[\text{dim } X])$

- **Theorem D:** [Huang-L-Zhu '14] There is a canonical isomorphism of D-modules
  
  $$\tau \equiv \tau(X, G) \simeq H^0\pi_+^\vee(\mathcal{O}_{V^\vee} \boxtimes D_{X,g})|_U.$$  

  As a consequence, for each $b \in V^\vee$, the solution space of $\tau$ at $b$ is given by

  $$H^n_c(U_b, \text{Sol}(D_{X,g})|_{U_b})$$

  the compactly supported cohomology of the *perverse sheaf*

  $$\text{Sol}(D_{X,g}) := R\text{Hom}_X(D_{X,g}, \mathcal{O}_X) \in D^b_c(X)^{op}$$

  restricted to $U_b \equiv X - Y_b$.

- This result describes $\tau$ as $D$-module in purely topological/cohomological terms, through perverse sheaves; the latter can be obtained by ‘gluing’ together locally constant sheaves on different strata of $X$.  

25. Consequences and applications

- Results and applications that we discussed so far, are all consequences of the cohomological description of $\tau$ (Theorem D.)

- If $X$ is $G$-homogeneous, then $D_{X, g}$ reduces to the structure sheaf, and Theorem D yields the completeness result (Theorem A).

- If $X$ is $T$-toric, then $D_{X, g}$ can be described in terms of certain rank 1 Kummer sheaves, and Theorem D yields the chain integral isomorphism (Theorem B).

- There are other cases where $D_{X, g}$ also allows a simple description. In these cases, the chain integral map can have kernel, but it can be described in purely geometric terms in terms of $G$-invariant subvarieties of $X$.

- There is a version of Theorem D for $\mathbb{Q}_\ell$ [Fu-Huang-L-Yau ’16], where $\text{Hom}(\tau, \mathcal{O}_{V^\vee})$ is replaced by a perverse sheaf in the derived category $D^b_c(\mathbb{V}^\vee, \mathbb{Q}_\ell)$. This generalizes results of [Fu ’12], who studied the special case $\tau = \tau_{GKZ}$, and gave a nice interpretation N. Katz’ hypergeometric sums including the classical Kloosterman sums. Our $\ell$-adic ‘tautological perverse sheaves’ describe further generalizations of Kloosterman sums in terms of period integrals.
Thank you.